Derivation of the Sum-of-Squares Relationship

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Abstract

The sum-of-squares relationship connects the square of elementary particle masses to the square of the Fermi scale. It constrains the spectrum of free parameters in the Standard Model (SM) and it suggests a straightforward resolution to the hierarchy problem. Here we show that this relationship follows from the minimal fractal structure of spacetime near the Fermi scale and supports the view that Dark Matter behaves as long-range multifractal replica of the SM.

Key words: Sum-of-squares, Standard Model, Minimal fractal manifold, Dark Matter, Cantor Dust.

1. Introduction and Background

The sum-of-square relationship of the SM links the square of elementary particle masses to the square of the Fermi scale viz.

$$m_W^2 + m_Z^2 + m_H^2 + \sum_f m_f^2 = \mathrm{v}^2 \tag{1}$$

where W, Z and H stand for the electroweak and the Higgs bosons, respectively, and the sum in the left-hand side is taken over the whole spectrum of SM fermions [1-2]. The contribution of bosons and fermions in (1) is split in nearly equal shares, that is,

$$\sum_b m_b^2 \approx \sum_f m_f^2 \approx \frac{\mathrm{v}^2}{2} \tag{2}$$

As conjectured in [2], (1) and (2) arise from the minimal fractal structure of spacetime near the Fermi scale, with (1) pointing to an unexpected resolution of the hierarchy problem. The goal here is to elaborate on this conjecture by appealing to concepts related to the geometry of multifractal Cantor set and to the attributes of random walks.

2. Multifractal Cantor Set in mass space

We proceed by introducing the concept of *mass space* M as an abstract Cartesian space formed by multiple mass coordinates. The spectrum of SM masses is entirely contained in its mass space, which is accordingly labeled $\mathrm{M}_{SM} \supset \{m_j\},\ j = 1, 2, ..., n$. Pursuing the framework of ideas put forward in [2, 8], the goal of this section is to uncover the dimensional description of M_{SM} as a *multifractal Cantor set*.

It is known that fractals are typically created starting from an elementary geometric object (the generator) and allowing its components ($i = 1, 2, ..., L$) to be independently scaled by a factor r_i, where $\sum r_i < 1$ [3]. Consider the simplest case of a Cantor set with two scales, $\mu_1 = m_1/\Delta < 1$ and $\mu_2 = m_2/\Delta < 1$. The recursive construction of the Cantor set consists of taking the segment of unit mass length, dividing it into segments of lengths $\mu_1, 1-(\mu_1+\mu_2), \mu_2$ and removing the middle segment. The division of segments continues indefinitely, generating a scale-reduced replica of the original construction. Aside from a scale factor, the subsets lying in the disjoint intervals $[0, \mu_1]$ and $[1-\mu_2, 1]$ are images of the whole set. Assume that the whole set is covered with unit segments (or *dimensional boxes*) of size δ. By definition, the number of boxes needed to cover the set is given by

$$N(\delta) \sim \delta^{-D_H} \tag{3}$$

in which D_H stands for the Hausdorff dimension. Upon magnification with scales $1/\mu_1$ and $1/\mu_2$, the number of boxes covering the interval $[0, \mu_1]$ and $[1-\mu_2, 1]$ are $N(\delta/\mu_1)$ and $N(\delta/\mu_2)$, respectively, which leads to

$$N(\delta) = N(\frac{\delta}{\mu_1}) + N(\frac{\delta}{\mu_2}) \tag{4}$$

or

$$\mu_1^{D_H} + \mu_2^{D_H} = 1 \tag{5}$$

The generalization of (5) when the generator is composed by Q elements is straightforward, namely,

$$\sum_{j=1}^{Q} \mu_j^{D_H} = 1 \tag{6}$$

3. Random walk in mass space and its Hausdorff dimension

Here we follow [3] in detail and assume that there is a probability distribution function underlying a random walk (RW) in $Q \geq 3$ dimensions of the mass space. It can typically take the form of a Gaussian or of a more general expression solely dependent on the magnitude of the mass coordinate,

$$p(\vec{\mu}) = p(|\vec{\mu}|) \tag{7}$$

We further confine the discussion to the case of RW's of fixed step size μ_0 and posit that the magnitude of the step is commensurate with the dimensional box introduced above (that is, $\mu_0 = O(\delta)$). In addition, we demand that all the moments of (7) stay finite. When $Q = 3$, (7) can be written as

$$p(\vec{\mu}) = \frac{1}{4\pi\mu_0^2}\delta_D(|\vec{\mu}| - \mu_0) \tag{8}$$

in which $\delta_D(...)$ stands for the Dirac delta function.

The explicit construction of the probability function starts with defining of a "coarse-grained" RW step

$$\vec{\mu}' = \sum_{k=1}^{n} \vec{\mu}_k \tag{9}$$

leading to

$$P(\vec{\mu}) = \int d^d\mu_1, ..., d^d\mu_n \, \delta_D(\vec{\mu} - \vec{\mu}')\, p(\vec{\mu}_1), ..., p(\vec{\mu}_n) \tag{10}$$

Introducing the Fourier transform of the delta function turns (10) into

$$P(\vec{\mu}') = \int \frac{d^d k}{(2\pi)^d} \exp(-i\vec{k}\cdot\vec{\mu}')\left[\left\langle \exp(i\vec{k}\cdot\vec{\mu}')\right\rangle\right]^n \tag{11}$$

in which the characteristic function of the probability distribution function $p(\vec{\mu})$ is given by

$$g(\vec{k}) = \left\langle \exp(i\vec{k}\cdot\vec{\mu}) \right\rangle = \int d^d\mu \exp(i\vec{k}\cdot\vec{\mu})\, p(\vec{\mu}) \tag{12}$$

One may invoke the cumulant expansion of the characteristic function, along with the symmetry condition $\left\langle \vec{\mu} \right\rangle = 0$, to show that the coarse-grained distribution (11) takes the form

$$P(\vec{\mu}') = \int \frac{d^d k}{(2\pi)^d} \exp(-i\vec{k}\cdot\vec{\mu}') \exp(-\frac{n}{2}\sigma_0^2 k^2 + \ldots) \tag{13}$$

where σ_0^2 is defined through the matrix of second moments $\sigma_{\omega\nu}^2$ as in

$$\sigma_{\omega\nu}^2 = \left\langle \mu_\omega \mu_\nu \right\rangle - \left\langle \mu_\omega \right\rangle \left\langle \mu_\nu \right\rangle = \sigma_0^2 \delta_{\omega\nu} \tag{14}$$

It can be also shown that the integrand of (13) is dominated by Fourier vectors k having magnitudes on the order of

$$k = O(\frac{1}{\sqrt{n\sigma_0^2}}) \tag{15}$$

Under these circumstances, (13) can be reduced to the familiar Gaussian distribution, namely,

$$P(\vec{\mu}') = \frac{1}{(2\pi n\sigma_0)^{d/2}} \exp(-\frac{\left|\vec{\mu}'\right|^2}{n\sigma_0^2}) \tag{16}$$

By symmetry, after n RW steps, the average "distance" in M_{SM} is $\left\langle \vec{\mu}' \right\rangle = 0$ and, since the individual steps are statistically uncorrelated, we have

$$\left\langle \overrightarrow{\mu_\omega} \cdot \overrightarrow{\mu_\nu} \right\rangle = 0 \tag{17}$$

for $\omega \neq \nu$. As a result,

$$\left\langle \left| \overrightarrow{\mu'} \right| \right\rangle^2 = \sum_{\omega=1}^{n} \sum_{\nu=1}^{n} \left\langle \overrightarrow{\mu_\omega} \cdot \overrightarrow{\mu_\nu} \right\rangle = \sum_{\omega=1}^{n} \left\langle \mu_\omega^2 \right\rangle = n\mu_0^2 \tag{18}$$

and the average root-mean-square distance amounts to

$$\mu = \sqrt{\left\langle \vec{\mu} \right\rangle^2} = \sqrt{n}\mu_0 = O(\sqrt{n}\delta) \tag{19}$$

If $n >> 1$, the RW may be partitioned into q sub-walks having n/q steps. Further assuming that $1 << q << n$, one can repeat the previous arguments for each of the sub-walks. By analogy with (19), the root-mean-square distance covered by each sub-walk is

$$\mu(q) = O(\sqrt{q}\delta) \tag{20}$$

Relations (19) and (20) indicate that the RW of n steps is *nearly self-similar* to the RW of n/q steps, each walk being defined by a length dependent on δ . Let $N(\delta)$ denote the number of dimensional boxes needed to cover the walk. Self-similarity arguments imply that $N(\delta)$ scales as

$$N(\delta) \sim \delta^{-2} = \delta^{-D_H} \tag{21}$$

which shows that the Hausdorff dimension of the RW is $D_H = 2$. Note that the same dimension applies for quantum mechanical paths, which hints to the underlying fractality of spacetime near or above the Fermi scale [10-12].

It is apparent that the sum-of-square relationship (1) can be directly recovered from the combined use of (6) and (21).

4. Cantor Dust as long-range replica of the SM

Recent research suggests that Dark Matter is a large-scale dimensional condensate of *Cantor Dust*, a topological structure emerging from the minimal fractal geometry of spacetime near or above the Fermi scale [4, 7, 9]. Remarkably, the Hausdorff dimension of Cantor Dust in three-dimensional space falls close to $D_H = 2$ and so is the Hausdorff dimensions of Penrose tiling used in modelling of atomic clusters and diffusion limited aggregation [5-7]. Taken together, these considerations favor the view that Dark Matter is a long-range replica of the SM and behaves like a superfluid phase on cosmic scales [9].

References

[1] https://arxiv.org/pdf/1305.4208.pdf

[2] Available at the following sites:

http://www.aracneeditrice.it/aracneweb/index.php/pubblicazione.html?item=9788854889972

https://www.researchgate.net/publication/278849474_Introduction_to_Fractional_Field_Theory_consolidated_version

[3] Creswick, R. J. *et al* "Introduction to Renormalization Group methods in Physics", John Wiley and sons, New York, 1992.

[4] https://www.prespacetime.com/index.php/pst/article/view/1563/1491

[5] https://en.wikipedia.org/wiki/List_of_fractals_by_Hausdorff_dimension

[6] http://www.iisc.ernet.in/currsci/aug102000/rc80.pdf

[7] Available at the following site:
https://www.researchgate.net/publication/329698623_Diffusion_Limited_Aggregation_and_the_Spiderweb_Distribution_of_Dark_Matter_on_Galactic_Scales

[8] Available at the following site:
https://www.ingentaconnect.com/content/asp/qm/2014/00000003/00000003/art00012?crawler=true

[9] https://prespacetime.com/index.php/pst/article/view/959/933

[10] http://inspirehep.net/record/143376/files/slac-pub-2427.pdf

[11] Available at the following site:
https://www.researchgate.net/publication/231155811_Fractal_space-time_A_geometric_analogue_of_relativistic_quantum_mechanics

[12] https://iopscience.iop.org/article/10.1088/0143-0807/11/6/004